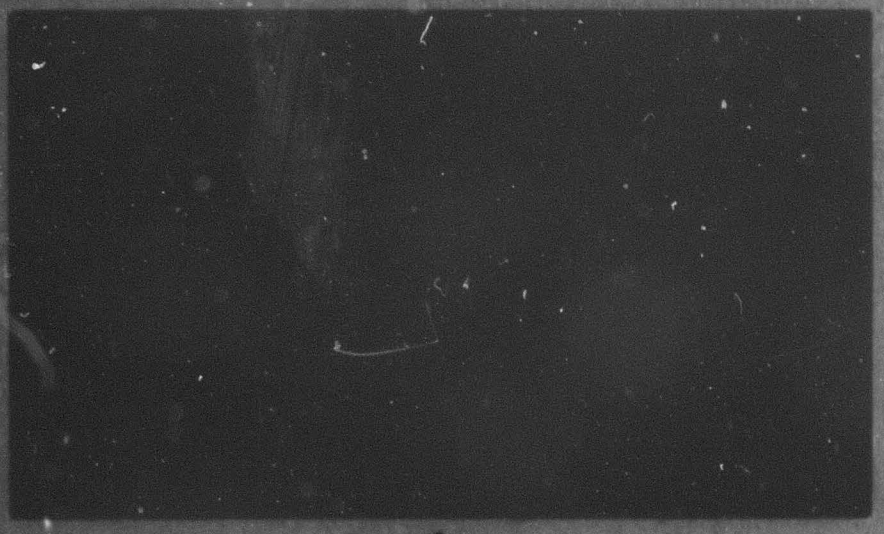
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CHARACTERIZATION OF THE KERNELS OF CONVEX GAMES

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ABSTRACT

A convex game is characterized by increasing marginal utility for coalition membership as coalitions grow larger. The core of any n-person game is the set of outcomes that cannot be profitably blocked by any coalition. It is known that various solution concepts bear direct relation to the core when the game concerned is convex. Thus, the von Neumann-Morgenstern solution and the bargaining set $\mathcal{M}_1^{(i)}$ coincide with the core and the value is essentially the center of gravity of the extreme points of the core (see [10],[6]). In this paper we consider another solution concept - the kernel, and show that for convex games it is a unique point, thus it coincides with the nucleolus of the game (see [9]), and constitutes another distinguished point of the core - different, in general, from the value. Roughly speaking, it is obtained by pushing inside at equal ℓ_1-distances certain supporting hyperplanes which determine the core, stopping the push of a hyperplane short of causing the inside to become void.

Introduction.

This paper can be regarded as a continuation of our study in [6]. In that paper we considered the kernels and the bargaining sets $\mathcal{M}_1^{(i)}$ (for the grand coalition) of convex games. We succeeded in characterizing the bargaining set, which, like the von-Neumann Morgenstern solution (see [14]), turned out to coincide with the core of the game. As to the kernel - we were only able to prove that it lies in the relative interior of the core; but we did not locate its exact position.

In this paper we prove that the kernel (for the grand coalition) of a convex game consists of a unique point and we characterize its location geometrically. Roughly speaking, the kernel is obtained by "pushing inside" at equal ℓ_1-distances certain hyperplanes which support the core, stopping the push of a hyperplane short of causing the inside to become empty. Thus, in general, the kernel differs from the Shapley value which is essentially the center of gravity of its core. (see [14]).

Every attempt has been made to render this paper self readable; yet, clearly, familiarity with the current literature is advisable.

In order to achieve our goals we had to extend known results about the kernels of monotonic games and to study the structure

of the core of convex games - subjects which are interesting in themselves. For this reason, when space considerations did not direct us otherwise, we stated and proved some of the theorems in more generality than needed for the specific goals as stated in the second paragraph of this introduction.

2. The Structure of the Pseudo Kernel for Monotonic Games.

Throughout most of this paper we shall be concerned with a cooperative monotonic n-person game with a non-negative characteristic function. Such a game will be denoted by $(N;v)$ where $N = \{1,2,\ldots,n\}$ is its set of players and its not necessarily superadditive characteristic function, by definition, satisfies*

$$(2.1) \qquad v(S) \geq 0 \qquad \text{all coalitions } S,\ S \subset N.$$

$$(2.2) \qquad v(S) \leq v(T) \qquad \text{whenever } S \subset T \subset N.$$

Note that (2.1) can be omitted if we introduce the empty coalition and define

$$(2.3) \qquad v(\emptyset) = 0.$$

Given an n-tuple $x = (x_1,x_2,\ldots,x_n)$ of real numbers, we define the excess of a coalition S with respect to x (in $(N;v)$) to be

$$(2.4) \qquad e(S,x) \equiv v(S) - x(S),$$

where $x(S)$ is a short notation for $\Sigma_{i\in S}\, x_i$ for $S \neq \emptyset$ and $x(\emptyset) = 0$.

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*The results of this and the next sections with the exception of Remark 3.5 will remain true if (2.2) is replaced by the weaker condition: $v(S) \leq v(T)$ whenever $S \subset T$ for $T \neq N$ (quasi monotonicity) and $v(\{i\}) + v(N) \geq v(S)$ for all $i \in N$ and $S \subset N - \{i\}$ (condition α) (See [5].)

An n-tuple $x = (x_1,x_2,\ldots,x_n)$ will be called a <u>pseudo-imputation</u>* (in $(N;v)$) if it satisfies

$$(2.5) \qquad x_i \geq 0, \qquad i = 1,2,\ldots,n,$$

$$(2.6) \qquad x(N) = v(N).$$

For each n-tuple $x = (x_1,x_2,\ldots,x_n)$ we define the <u>maximum surplus</u> of a player k against a player ℓ, $k \neq \ell$, with respect to x to be

$$(2.7) \qquad s_{k,\ell}(x) \equiv \max_{S:k\in S,\ell\notin S} e(S,x).$$

<u>Definition 2.1</u>.** Let $\Gamma \equiv (N;v)$ be a cooperative n-person game whose characteristic function satisfies (2.1). A pseudo-imputation x is said to belong to the <u>pseudo-kernel</u> of Γ (for the grand coalition), if

$$(2.8) \qquad s_{k,\ell}(x) \leq s_{\ell,k}(x) \quad \text{or} \quad x_\ell = 0 \quad \text{for all } k,\ell\in N,\ k \neq \ell.$$

The pseudo-kernel of Γ (for the grand coalition) will be denoted by $\mathcal{PK}(\Gamma)$ or, shortly, by $\mathcal{PK}$. It is known that it is never empty (see [4],[5]).

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*The term "pseudo" comes to denote that the usual individual-rationality requirement is replaced by the weaker condition (2.5).

**The definition can be extended to cover situations in which coalition structures other than the grand coalition are formed (see [2]).

It is proved in [5]* that if Γ satisfies the monotonicity condition (2.3) in addition, then "or $x_\ell = 0$" in (2.8) is redundant and (2.8) simply becomes

$$(2.8)_a \quad s_{k,\ell}(x) = s_{\ell,k}(x) \quad \text{for all } k,\ \ell,\ k \neq \ell.$$

Relations (2.7) - (2.8) show that the pseudo-kernel is a finite union of closed convex polyhedra. In this and the next section we shall characterize them and derive properties concerning the structure of the pseudo-kernel which somewhat generalize results stated in [4], [5] for the special class of games satisfying (2.2), (2.3).**

Let $\Gamma \equiv (N;v)$ be a game satisfying (2.2), (2.3) and let x be an n-tuple of real numbers. We can partition the set of coalitions into subsets $\mathcal{C}^1(x)$, $\mathcal{C}^2(x),\ldots,\mathcal{C}^m(x)$ which are of the highest excess, of the second highest excess, etc. Thus,

$$(2.9) \quad \mathcal{C}^1(x) \equiv \{S:\ e(S,x) \geq e(T,x) \quad \text{all } T\}$$

$$(2.10) \quad \mathcal{C}^{i+1}(x) \equiv \{S:\ S \notin \mathcal{C}^1(x) \cup \mathcal{C}^2(x) \cup \ldots \cup \mathcal{C}^i(x) \text{ and } e(S,x) \geq e(T,x) \text{ whenever } T \notin \mathcal{C}^1(x) \cup \mathcal{C}^2(x) \cup \ldots \cup \mathcal{C}^i(x)\},$$

$i = 1,2,\ldots m$ and $m = m(x)$ is the highest index i for which $\mathcal{C}^i(x)$ is not empty. Clearly, $1 \leq m \leq 2^n$.

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*Corollary 3.9 in [5]. Note that K in this corollary as well as ir the proof is wrong. It should be replaced by PK.

**Similar generalizations for arbitrary games are straightforward but longer to state. At any rate, they are not needed in this paper.

We shall refer to the coalitions in $\mathcal{E}^i(x)$ as the i-th stage maximum excess coalitions. Their excess $s^i(x)$ will be called the i-th stage maximum excess.

(2.11) $s^i(x) = e(S,x)$ where $S \in \mathcal{E}^i(x)$.

Given that x is a pseudo-imputation, it is possible to tell from the sets $\mathcal{E}^i(x)$ whether $x \in PK(\Gamma)$. Indeed, let us define

(2.12) $i(k,\ell,x) \equiv \text{Min}\{i: \exists\, S \in \mathcal{E}^i(x),\ k \in S,\ \ell \notin S\}$.

Clearly,

(2.13) $s_{k,\ell}(x) = s^{i(k,\ell,x)}(x)$

(see (2.7)). The following lemma follows from (2.8) and (2.13).

LEMMA 2.2: Let $\Gamma \equiv (N;v)$ be a monotonic game having a non-negative characteristic function. A pseudo imputation x belongs to $PK(\Gamma)$ if and only if $i(k,\ell,x) = i(\ell,k,x)$ for each pair of distinct players k and ℓ.

We can now reverse the procedure. Consider an arbitrary vectorial partition* $(\mathcal{E}^1, \mathcal{E}^2, \ldots, \mathcal{E}^m)$ of the set of all the coalitions which has the property

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*It is important to remember which coalitions belong to what stages. Thus, for $N = \{1,2,3\}$ we consider $(\{\emptyset,N\}, \{\{1,2\},\{1,3\},\{2,3\}\}, \{\{1\},\{2\},\{3\}\})$ to be different from $(\{\emptyset,N\}, \{\{1\},\{2\},\{3\}\}, \{\{1,2\}, \{1,3\}, \{2,3\}\})$. This is why we use the vector notation and call the partition vectorial.

(2.14) $i(k,\ell) = i(\ell,k)$ for all $\ell,k \in N$, $\ell \neq k$,

where

(2.15) $i(k,\ell) = \text{Min}\{i: \exists\, S \in \mathcal{C}^i,\ k \in S,\ \ell \notin S\}$.

Every pseudo-imputation x satisfying

(2.16) $\mathcal{C}^i(x) = \mathcal{C}^i, \qquad i = 1,2,\ldots,m,$

must belong to $PK(\Gamma)$. Moreover, by scanning all possible vectorial partitions, one obtains all the points of the pseudo-kernel.

Observe that the set of pseudo-imputations satisfying (2.16) for a fixed vectorial partition is a convex polyhedron. Indeed, its closure is the solution of the system of weak linear inequalities

$$(2.17)\quad \begin{cases} x(N) = v(N), \\ x_i \geq 0, \quad i = 1,2,\ldots,n, \\ e(S,x) \geq e(T,x) \text{ whenever } S \in \mathcal{C}^\mu,\ T \in \mathcal{C}^\nu,\ \mu \leq \nu. \end{cases}$$

provided that it is not empty.

Our next object, therefore, is to find conditions which assure us that a vectorial partition satisfies (2.14). The following definition will be quite helpful:

Let $\mathcal{C}$ be a collection of subsets of N and let T be a fixed subset of N. Let $\{T_1,T_2,\ldots,T_u\}$ be the partition of T characterized by:

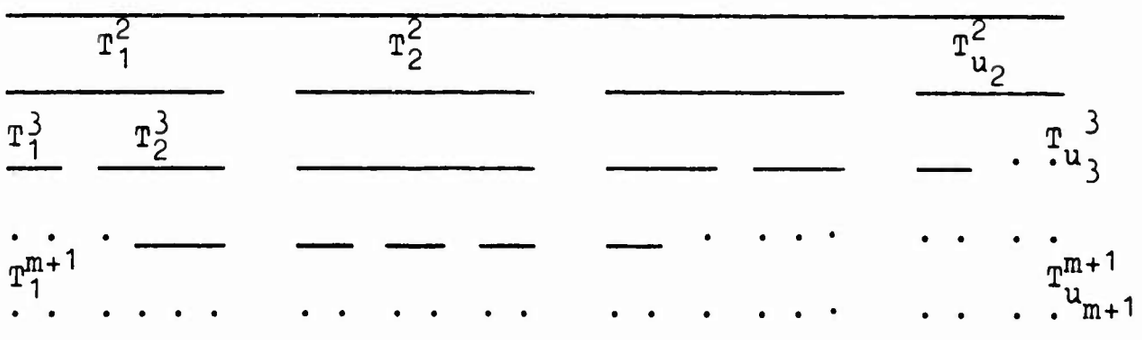
(2.18) $k,\ell \in T_j \iff (k,\ell \in T$ and $k \in A$ if and only if $\ell \in A$ for all $A \in \mathcal{C})$.

Thus, the T_j's are equivalence classes under the relation "occur simultaneously in the coalitions of $\mathcal{C}$ ".

Definition 2.3: The set $\{T_1, T_2, \ldots, T_u\}$ defined by (2.18) will be called the partition of T into equivalence classes induced by $\mathcal{C}$.

Let $E \equiv (\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^m)$ be an arbitrary vectorial partition of the set of all the coalitions among members of N. We shall construct by induction a profile $P(E)$ generated by E:

We start by denoting $\{N\}$ also as $\{T_1^1\}$. Suppose that $\{T_1^i, T_2^i, \ldots, T_{u_i}^i\}$ has been defined, and it is a partition of N. Let $\{T_{j,1}^{i+1}, T_{j,2}^{i+1}, \ldots, T_{j,\alpha_j}^{i+1}\}$ be the set of equivalence classes which are induced by $\mathcal{C}^i$ on T_j^i, $j = 1,2,\ldots,u_i$. Renumber the $T_{j,\nu}^{i+1}$ lexicographically in the lower indices to form $\{T_1^{i+1}, T_2^{i+1}, \ldots, T_{u_{i+1}}^{i+1}\}$. The collection $P(E) \equiv \{T_1^1; T_1^2, \ldots, T_{u_2}^2; T_1^{m+1}, \ldots, T_{u_{m+1}}^{m+1}\}$ will be called the profile generated by E. The term is suggested by the diagram below.



Clearly,

(2.19) $\{T_1^{m+1},\ldots,T_{u_{m+1}}^{m+1}\} = \{\{1\},\{2\},\ldots,\{n\}\}$,

and, in general, the equivalence classes will become 1-person sets also for stages with a smaller index.

Lemmas 2.4 and 2.5 follow directly from the definitions.

LEMMA 2.4: If $1 \le i_0 \le i_1 \le m+1$, then

(2.20) $T_{j_0}^{i_0} \cap T_{j_1}^{i_1} \neq \emptyset$ implies $T_{j_1}^{i_1} \subset T_{j_0}^{i_0}$.

LEMMA 2.5: If $S \in \mathcal{C}^i$ then S is a union of sets T_j^{i+1}'s.

LEMMA 2.6: If $S \in \mathcal{C}^{i_0}$ then S is a union of sets $T_j^{i_1}$'s whenever $i_0 < i_1 \le m+1$.

PROOF: This is a consequence of the previous two lemmas and from the fact that $\{T_1^i,\ldots,T_{u_i}^i\}$ is a partition of N.

Henceforth, the profile $\mathcal{P}(\boldsymbol{\mathcal{E}}(x))$ generated by the partition $\boldsymbol{\mathcal{E}}(x) \equiv (\mathcal{C}^1(x),\mathcal{C}^2(x),\ldots,\mathcal{C}^m(x))$ will be called, shortly, the profile of x.

Lemmas 2.4 - 2.5 indicate that the profile can be described as a "partition tree"; namely as a tree whose vertices are the sets T_j^i, $T_1^1 \equiv N$ being its root, such that the vertices that are below a vertex T_j^i and are adjacent to it form a partition of T_j^i.

One of the advantages of the profile $P(\mathcal{E})$ of a vectorial partition $\mathcal{E}$ of the set of coalitions is the fact that it enables us to state the condition (2.14) in a more visual fashion.

THEOREM 2.7: Let $\mathcal{E} = \{\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^m\}$ be a vectorial partition of the set of coalitions in N. Let $P(\mathcal{E})$ be the profile generated by $\mathcal{E}$. The condition (2.14) is equivalent to the following condition:

The separation condition. If $T_k^{i+1} \subset T_j^i$, $T_\ell^{i+1} \subset T_j^i$ and $k \neq \ell$, then there exists a coalition S in $\mathcal{C}^i$ such $T_k^{i+1} \subset S$ and $T_\ell^{i+1} \cap S = \emptyset$.

PROOF: Suppose (2.14) is satisfied. Let $T_k^{i_0+1}$, $T_\ell^{i_0+1}$ be subsets of $T_j^{i_0}$. Consider a player in $T_k^{i_0+1}$ and a player in $T_\ell^{i_0+1}$ whom, without loss of generality, we call players k and ℓ, respectively. Clearly, $k \in A \iff \ell \in A$, whenever $A \in \mathcal{C}^i$ and $i \leq i_0 - 1$ (Lemma 2.4). Since k and ℓ belong to different equivalence classes of the stage $i_0 + 1$, either there exists a coalition in $\mathcal{C}^{i_0}$, containing k and not containing ℓ, or there exists a coalition in $\mathcal{C}^{i_0}$ containing ℓ and not containing k. By (2.15), either $i(k,\ell) = i_0$ or $i(\ell,k) = i_0$; hence, by (2.14), $i(k,\ell) = i_0$. Consequently, by (2.15), there exists a coalition S in $\mathcal{C}^{i_0}$ containing k and not containing ℓ. By Lemma 2.5, $S \supset T_k^{i_0+1}$ and $T_\ell^{i_0+1} \cap S = \emptyset$; therefore, the separation condition is satisfied.

Conversely, suppose that the separation condition is satisfied. Let k and ℓ be two distinct players. There exists an index i_0 such that $k,\ell \in T_j^{i_0}$ and k,ℓ belong to two distinct equivalence classes of the stage $i_0 + 1$ which, without loss of generality, we name $T_k^{i_0+1}$ and $T_\ell^{i_0+1}$, respectively. By the separation condition, there exist coalitions R and S in $\mathcal{C}^{i_0}$ such that $k \in R$, $\ell \notin R$, $\ell \in S$, $k \notin S$. By Lemma 2.4, $k \in A \Longleftrightarrow \ell \in A$ whenever $A \in \mathcal{C}^i$, $i \leq i_0 - 1$. Consequently, by (2.15), $i(k,\ell) = i(\ell,k) = i_0$ and (2.14) is satisfied. This completes the proof.

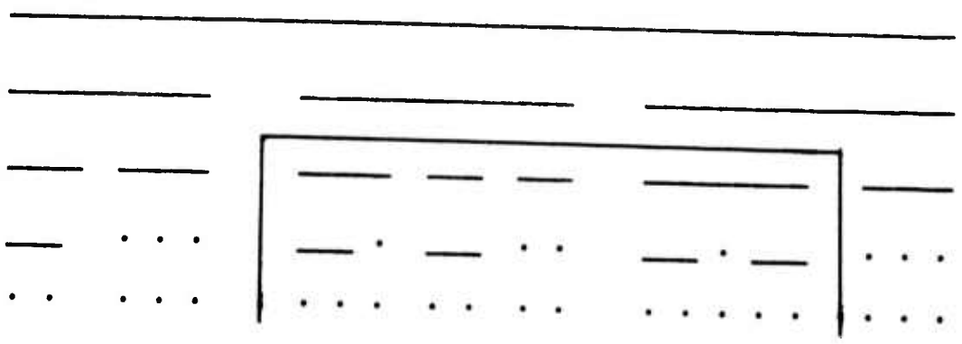
COROLLARY 2.8: Let $\Gamma \equiv (N;v)$ be a monotonic game having a non-negative characteristic function. Let

$$P(\mathcal{E}(x)) \equiv \{T_1^1;\ T_1^2,\ T_2^2,\ldots,\ T_{u_2}^2;\ldots;T_1^{m+1},T_2^{m+1},\ldots,T_{u_{m+1}}^{m+1}\}$$

be the profile of a pseudo-imputation x. With this notation, $x \in PK(\Gamma)$ if and only if the separation condition in Theorem 2.7 with $\mathcal{C}^i = \mathcal{C}^i(x)$ is satisfied, $i = 1,2,\ldots,m$.

3. The Stage Game.

From a visual point of view, a profile may contain smaller profiles. The figure below exhibits one profile within the original one.



This suggests that smaller games can be constructed from the original game, which contain fewer players. Such games can serve for induction purposes.

If one examines Theorem 2.7, one finds that only the equivalence classes play a role in each stage and not the individual players. Even the maximum excess coalitions of the various stages are unions of such equivalence classes (Lemma 2.5). This suggests that it is possible under an appropriate interpretation to regard the equivalence classes themselves as players in some certain games. In the present section we shall develop these heuristic ideas in a precise way.

Definition 3.1: Let $\Gamma \equiv (N;v)$ be a monotonic game with a non-negative characteristic function. Let x be a non-negative

n-tuple and let $P(E(x)) = \{T_1^1;\ T_1^2,\ldots,\ T_{u_2}^2;\ldots;\ T_1^{m+1},\ldots,T_{u_{m+1}}^{m+1}\}$ be the profile of x.

Let $T^* = \{T_{j_1}^i,\ T_{j_2}^i,\ldots,T_{j_\alpha}^i\}$ be an arbitrary set of equivalence classes belonging to a fixed stage i. The stage game generated by x and T^* is a game $(T^*;\ v^*)$ whose players are the members of T^* and its characteristic function is defined by

$$(3.1)\quad \begin{cases} v^*(T^*) = x(T_{j_1}^i) + x(T_{j_2}^i) + \ldots + x(T_{j_\alpha}^i) = x(T) \\ v^*(S^*) = \underset{Q:Q \subset N-T}{\text{Max}}\ v(S \cup Q) - x(Q),\ \emptyset \neq S^* \subset T^*, S^* \neq T^* \\ v^*(\emptyset) = 0. \end{cases}$$

Here, $T \equiv T_{j_1}^i \cup \ldots \cup T_{j_\alpha}^i$ and if $S^* = \{T_{\nu_1}^i,\ldots,T_{\nu_\beta}^i\} \subset T^*$, then $S = T_{\nu_1}^i \cup \ldots \cup T_{\nu_\beta}^i$.

Remark 3.2: The stage game $(T^*;v^*)$ satisfies (2.1) and (2.3). It is also quasi monotonic (see first footnote in Section 2.)

Definition 3.3: A pseudo imputation x is said to belong to the core of the game if,

$$(3.2)\quad e(S,x) \leq 0 \qquad \text{all } S.$$

Remark 3.4: A pseudo imputation which belongs to the core is an imputation (see second footnote in Section 2.)

PROOF: Individual rationality is nothing but (3.2) restricted to single-person coalitions.

Remark 3.5: The stage game $(T^*;v^*)$ is monotonic if x belongs to the core of the game.*

We are now in a position to state the main theorem of this section:

THEOREM 3.6: Let $\Gamma \equiv (N;v)$ be a monotonic game having a non-negative characteristic function. Let $x \in PK(\Gamma)$ and let $\Gamma^* \equiv (T^*;v^*)$ be a stage game generated by x and a set $T^* = \{T^i_{j_1},\ldots, T^i_{j_\alpha}\}$ of equivalence classes of an i-th stage, $1 \le i \le m+1$. Denote by x^* the α-tuple $x^* \equiv (x(T^i_{j_1}),\ldots,x(T^i_{j_\alpha}))$.

Under these conditions, x^* belongs to $PK(\Gamma^*)$.

PROOF: We shall use stars to denote entities related to the game Γ^*. Clearly, x^* is a pseudo-imputation in Γ^* (see (3.1)). Consequently, there is nothing more to prove if $\alpha = 1$. Suppose $\alpha > 1$. Since we do not know if Γ^* is monotonic, we have to use a starred analogue of (2.8) for the definition of its pseudo-kernel. However, since every pseudo-imputation satisfying $(2.8)_a$ automatically satisfies (2.8), our proof will be completed if we show that x^* satisfies

$$(3.3) \qquad s^*_{T^i_\rho, T^i_\sigma}(x^*) = s^*_{T^i_\sigma, T^i_\rho}(x^*)$$

---

*This result does not necessarily hold if, instead of being monotonic, Γ is quasi monotonic and satisfies condition α (see first footnote in Section 2).

for all pairs (ρ,σ), $\rho \neq \sigma$, $\rho,\sigma \in \{j_1,\ldots,j_\alpha\}$. Here,

$$(3.4)\qquad s^*_{T^i_\rho,T^i_\sigma}(x^*) \equiv \operatorname{Max}\{e^*(S^*,x^*):$$

$$S^* = \{T^i_{\nu_1},\ldots,T^i_{\nu_\beta}\},\ \{\nu_1,\ldots,\nu_\beta\} \subset \{j_1,\ldots,j_\alpha\},$$

$$\rho \in \{\nu_1,\ldots,\nu_\beta\},\ \sigma \notin \{\nu_1,\ldots,\nu_\beta\}\ \},$$

and

$$(3.5)\qquad e^*(S^*,x^*) \equiv v^*(S^*) - \sum_{\mu=1}^{\beta} x^*(T^i_{\nu_\mu})$$

if $S^* = \{T^i_{\nu_1},\ldots,T^i_{\nu_\beta}\}$.

By (3.4) and (3.1),

$$s^*_{T^i_\rho,T^i_\sigma}(x^*) = \operatorname{Max}\{\operatorname*{Max}_{Q:Q\subset N-T} e(T^i_{\nu_1}\cup\ldots\cup T^i_{\nu_\beta}\cup Q,x):$$

$$\rho \in \{\nu_1,\ldots,\nu_\beta\} \subset \{j_1,\ldots,j_\alpha\},\ \sigma \notin \{\nu_1,\ldots,\nu_\beta\}\ \} =$$

$$= \operatorname{Max}\{e(S,x): k\in S,\ \ell\notin S\} \equiv s_{k,\ell}(x),$$

where k is any player in T^i_ρ and ℓ is any player in T^i_σ. The argument for the validity of the last equality (not counting the identity sign) runs as follows: A priori, there should be there an inequality sign $\leq$, because the set of candidates increases. By (2.13), $s_{k,\ell}(x) = s^{i(k,\ell,x)}(x)$. Moreover, since T^i_ρ and T^i_σ are distinct equivalence classes of the i-th stage and $k\in T^i_\rho$ $\ell\in T^i_\sigma$, it follows that in the profile of x, k and ℓ belong the first time to two disjoint equivalence classes at a stage not later than i. Since $x\in P_K(\Gamma)$, it follows from

Lemma 2.2, that $i(k,\ell,x) \leq i - 1$. Let S^o be a coalition containing k and not containing ℓ such that $s_{k,\ell}(x) = e(S^o,x)$. By (2.7), $S^o \in \mathcal{E}^{i(k,\ell,x)}(x)$. Since $i(k,\ell,x) \leq i - 1$, it follows from Lemma 2.6 that S^o is a union of equivalence classes of the i-th stage and, moreover, $S^o \supset T^i_\rho$, $S^o \cap T^i_\sigma = \emptyset$.

Thus, $S^o = T^i_{\nu_1} \cup \ldots \cup T^i_{\nu_\beta} \cup Q$ where $\{\nu_1,\ldots,\nu_\beta\}$ is a subset of $\{j_1,\ldots,j_\alpha\}$ containing ρ and not containing σ and $Q \subset N-T$. It is therefore a member of the smaller set of candidates, which proves the equality mentioned above. In a similar fashion we prove that $s^*_{T^i_\sigma, T^i_\rho}(x^*) = s_{\ell,k}(x)$ and since $x \in \mathcal{P}K(\Gamma)$ it now follows that $x^* \in \mathcal{P}K(\Gamma^*)$.

Remark 3.7: If T^* is the set of all the equivalence classes of stage $m + 1$ and x is a pseudo imputation, then, by (3.1), the stage game $(T^*; v^*)$ is isomorphic to the game $(N;v)$. The transformation $\{k\} \longleftrightarrow k$ leads from one game to the other. Thus, the converse of Theorem 3.6, namely, $x^* \in \mathcal{P}K(\Gamma^*)$ for every stage game (and, in particular, the $(m + 1)$-th stage game implies $x \in \mathcal{P}K(\Gamma)$ is trivially true.

Remark 3.8: Theorem 3.6 generalizes results of [5]. The stage game in which T^* consists of all the equivalence classes of a given stage is known as an intermediate game. The stage game in which $i = m + 1$ and $\{j_1,\ldots,j_\alpha\}$ are players of a given $T^{i_o}_j$ is known as a reduced game.

4. The Stage Games resulting from an imputation in the core of a convex game.

A cooperative game $(N;v)$ is called convex if its characteristic function v satisfies

$$(4.1) \qquad v(\emptyset) = 0$$

$$(4.2) \qquad v(A) + v(B) \leq v(A \cup B) + v(A \cap B) \qquad \text{all } A,B \subset N.$$

Convex games were introduced in [14], where their properties, their importance in game theory and applications were discussed. In particular, it was shown that such games have non-empty cores.* In [6] we proved that the kernel for the grand coalition of a convex game is contained in the relative interior of the core. In this section we shall show that for an x in the core of a convex game, all the stage games are also convex. We shall also study the nature of some of these stage-games.

Convex games are super-additive but not necessarily monotonic. However, if the characteristic function satisfies

$$(4.3) \qquad v(\{i\}) \geq 0, \qquad i = 1,2,\ldots,n,$$

then monotonicity follows from super-additivity. In particular, (2.1) follows from (4.1) - (4.3).

Note that (4.2) is equivalent to

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*Moreover, they are characterized by the fact that their cores are regular.

(4.4) $\quad e(A,x) + e(B,x) \leq e(A \cup B,x) + e(A \cap B,x)$

for all $A, B \subset N$ and for any arbitrarily chosen fixed n-tuple x.

THEOREM 4.1: If $\Gamma \equiv (N;v)$ is a convex game and x belongs to its core, then each stage game generated by x is convex. If, furthermore, Γ has a non-negative characteristic function then this property passes on to the stage games.

PROOF: Let $(T^*;v^*)$ be a stage game generated by x. Since x is in the core it follows that (3.1) is equivalent to

$$(4.5)\quad \begin{cases} v^*(T^*) = x(T^i_{j_1}) + x(T^i_{j_2}) + \ldots + x(T^i_{j_\alpha}) \\ v^*(S^*) = \underset{Q:Q \subset N-T}{\text{Max}} [v(S \cup Q) - x(Q)], \quad S^* \subset T^*, \ S^* \neq T^*, \end{cases}$$

where $T = T^i_{j_1} \cup \ldots \cup T^i_{j_\alpha}$ and if $S^* = \{T^i_{\nu_1}, \ldots, T^i_{\nu_\beta}\}$, $\{\nu_1, \ldots, \nu_\beta\} \subset \{j_1, \ldots, j_\alpha\}$, then $S = T^i_{\nu_1} \cup \ldots \cup T^i_{\nu_\beta}$. Clearly, if $v(S) \geq 0$ for each coalition S then $v^*(S^*) \geq 0$ whenever $S^* \subset T^*$. It remains to show that

(4.6) $\quad v^*(S^*) + v^*(R^*) \leq v^*(S^* \cup R^*) + v^*(S^* \cap R^*)$ all $S^*, R^* \subset T^*$.

Relation (4.6) evidently holds if $S^* \subset R^*$ or if $R^* \subset S^*$. We can therefore assume that $S^*, R^* \neq T^*$. Let S and R be the unions of the members of S^* and R^*, respectively. By (4.5), there exist Q_1 and Q_2 in N-T such that $v^*(S^*) +$

$+ v^*(R^*) = v(S \cup Q_1) - x(Q_1) + v(R \cup Q_2) - x(Q_2) \leq v((S \cup R) \cup (Q_1 \cup Q_2)) +$

$$+ v((S\cap R) \cup (Q_1\cap Q_2)) - x(Q_1\cup Q_2) - x(Q_1\cap Q_2) \leq \max_{Q:Q\subset N-T} [v((S\cup R)\cup Q) -$$

$$- x(Q)] + \max_{Q:Q\subset N-T} [v((S\cap R)\cup Q) - x(Q)] = \max_{Q:Q\subset N-T} (v((S\cup R)\cup Q -$$

$$- x(Q)] + v^*(S^* \cap R^*).$$

If $S^*\cup R^* = T^*$ then

$$(4.7) \qquad \max_{Q:Q\subset N-T} [v((S\cup R)\cup Q) - x(Q)] \leq x(T^i_{j_1}) + \ldots + x(T^i_{j_\alpha}) =$$

$$= v^*(S^*\cup R^*),$$

because x belongs to the core of Γ. In any case (4.6) holds.

The following lemma furnishes much information concerning particular Q's for which the maxima in (4.5) are achieved.

LEMMA 4.2: Let $\Gamma \equiv (N;v)$ be a convex game and let x be an arbitrary n-tuple. Let R be a coalition in $\mathcal{C}^i(x)$ and let S_1, S_2 be subsets of R and $N-R$, respectively. Suppose that Q_1 and Q_2 are subsets of $N-R$ and R, respectively, such that

$$(4.8) \qquad \max_{Q:Q\subset N-R} e(S_1\cup Q,x) = e(S_1\cup Q_1,x)$$

$$(4.9) \qquad \max_{Q:Q\subset R} e(S_2\cup Q,x) = e(S_2\cup Q_2,x).$$

Let $R\cup Q_1$ and Q_2 belong to $\mathcal{C}^{\mu_1}(x)$ and $\mathcal{C}^{\mu_2}(x)$, respectively. Under these conditions

(i) $\mu_1 \leq i$,

(ii) $\mu_2 \leq i$,

(iii) If $e(S_1 \cup Q_1,x) \neq e(S_1,x)$ then $\mu_1 < i$,

(iv) If $e(S_2 \cup Q_2,x) \neq e(S_2 \cup R,x)$ then $\mu_2 < i$.

PROOF: By (4.4),

$$(4.10) \quad e(S_1 \cup Q_1, x) + e(R, x) \leq e(R \cup Q_1, x) + e(S_1, x).$$

By definition, $e(S_1 \cup Q_1, x) \geq e(S_1, x)$ and strong inequality holds if the condition in (iii) is satisfied. Consequently,

$$(4.11) \quad e(R, x) \leq e(R \cup Q_1, x)$$

and strong inequality holds if the condition in (iii) is satisfied. This proves (i) and (iii).

By (4.4),

$$(4.12) \quad e(S_2 \cup Q_2, x) + e(R, x) \leq e(S_2 \cup R, x) + e(Q_2, x).$$

By definition, $e(S_2 \cup Q_2, x) \geq e(S_2 \cup R)$ and strong inequality holds if the condition in (iv) is satisfied. Consequently,

$$(4.13) \quad e(R, x) \leq e(Q_2, x)$$

and strong inequality holds if the condition in (iv) is satisfied. This proves (ii) and (iv).

COROLLARY 4.3: Q_1 and Q_2 of Lemma 4.2 can be chosen to be unions of equivalence classes of stage $i + 1$ in the profile of x.

PROOF: This follows from cases (i) and (ii) of Lemma 4.2 and from Lemma 2.6.

COROLLARY 4.4: If $R \in \mathcal{E}^1(x)$ then

$$(4.14) \quad \max_{Q : Q \subset N - R} e(S \cup Q, x) = e(S, x) \quad \text{whenever } S \subset R.$$

(4.15) $\underset{Q:Q \subset R}{\text{Max}}\ e(SUQ,x) = e(SUR,x)$ <u>whenever</u> $S \subset N-R$.

<u>PROOF</u>: Cases (iii) and (iv) of Lemma 4.2.

Lemma 4.2 can be effectively used in devising programs for computing the pseudo-kernels (and the kernels) for convex games. Note that it can be applied to any stage-game $(T^*;v^*)$ of a stage greater than i, when the union of the members of T^* is equal to R. We shall subsequently apply the lemma for the particular cases $i = 1,2$ and the stage-games being the stage $m + 1$.

We shall now add the assumption that x belongs to the core of the game, and treat the collection $\mathcal{D}(x)$ defined by

(4.16) $\mathcal{D}(x) = \{S:S \neq \emptyset,N$ and $e(S,x) \geq e(R,x)$ whenever $R \neq \emptyset,N\}$.

<u>LEMMA 4.5</u>: <u>If</u> x <u>belongs to the core of a game</u> $\Gamma \equiv (N;v)$ <u>then either</u> $\mathcal{D}(x) \cup \{\emptyset\} \cup \{N\} = \mathcal{E}^1(x)$ or $\mathcal{D}(x) = \mathcal{E}^2(x)$, <u>in which case</u> $\mathcal{E}^1(x) = \{\emptyset, N\}$. <u>In the first case</u> $\mathcal{D}(x)$ <u>induces on N the equivalence classes of the second stage of the profile generated by</u> x; <u>in the second case it induces on N the equivalence classes of the third stage of the profile generated by</u> x, <u>the second stage being</u> $\{N\}$.

The proof is straightforward.

<u>LEMMA 4.6</u>: <u>Let</u> $\Gamma \equiv (N,v)$ <u>be a convex game and let</u> x <u>belong to the core of</u> Γ. <u>If</u> $R \in \mathcal{D}(x)$ <u>then</u>

$$(4.17)\qquad \underset{Q:Q \subset N-R}{\text{Max}} \; e(SUQ,x) = \text{Max}\,[e(S,x),\ e(S \cup (N-R),x)]$$

whenever $S \subset R$,

$$(4.18)\qquad \underset{Q:Q \subset R}{\text{Max}} \; e(SUQ,x) = \text{Max}\,[e(S,x),\ e(SUR,x)]$$

whenever $S \subset N-R$.

PROOF: Corollary 4.4, if $\mathcal{D}(x) \subset \mathcal{E}^1(x)$. If this is not the case then, by Lemma 4.5, $\mathcal{E}^1(x) = \{\emptyset, N\}$ and $\mathcal{D}(x) = \mathcal{E}^2(x)$. The result now follows from Lemma 4.2.

COROLLARY 4.7: Let $\Gamma \equiv (N;v)$ be a convex game and let x belong to its core. Let R be a coalition in $\mathcal{D}(x)$ and consider the stage games $(T_R^*;\ v_R^*)$ and $(T_{N-R}^*;\ v_{N-R}^*)$ of any stage i, such that the union of the members of T_R^* is equal to R and the union of the members of T_{N-R}^* is $N-R$. Under these conditions

$$(4.19)\begin{cases} v_R^*(T_R^*) = x(R) \\ v_R^*(S^*) = \text{Max}\,[v(S),\ v(S \cup (N-R)) - x(N-R)], \\ \qquad\qquad T_R^* \neq S^* \subset T^*{}_R \end{cases}$$

$$(4.20)\begin{cases} v_{N-R}^*(T_{N-R}^*) = x(N-R) \\ v_{N-R}^*(S^*) = \text{Max}\,[v(S),\ v(SUR) - x(R)], \\ \qquad\qquad T_{N-R}^* \neq S^* \subset T_{N-R}^*. \end{cases}$$

Here, $S = T_{\mu_1}^i \cup T_{\mu_2}^i \cup \ldots \cup T_{\mu_\beta}^i$ if $S^* = \{T_{\mu_1}^i, T_{\mu_2}^i, \ldots, T_{\mu_\beta}^i\}$.

5. Completely Separating Near Ring Collections.

Definition 5.1. A collection $\mathcal{C}$ of subsets of a set N is called a near-ring* if,

(5.1) $A,B \in \mathcal{C} \Rightarrow A \cup B = N$ or $A \cap B = \emptyset$ or both $A \cup B$ and $A \cap B$ belong to $\mathcal{C}$.

LEMMA 5.2: If Γ is a convex game and x is an arbitrary n-tuple then $\mathcal{D}(x)$ (see (4.16) is a near-ring.

PROOF: Combine (4.4) with (4.16).

Definition 5.3: A collection $\mathcal{C}$ of subsets of a set N is said to be completely separating (over N), if for each ordered pair (k,ℓ) of elements of N, $k \neq \ell$, there exists a set in $\mathcal{C}$ containing k and not containing ℓ.

Definition 5.4: A collection $\mathcal{C}$ of subsets of a set N is called separating (over N), if for each ordered pair (k,ℓ) of elements of N, if a coalition exists in $\mathcal{C}$, which contains k and does not contain ℓ, then another coalition exists in $\mathcal{C}$ which contains ℓ and does not contain k.

Let $\mathcal{C}$ be a separating collection of subsets of N. Let $T_1, T_2, \ldots, T_u$ be the equivalence classes induced by

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*We are grateful to J. R. Isbell for suggesting this term.

$\mathcal{E}$ on N. Let $N^{\sim}$ be a subset of N containing exactly one member from each equivalence class. Clearly, the collection $\mathcal{E}^{\sim} \equiv \{S \cap N^{\sim} | S \in \mathcal{E}\}$ is completely separating over $N^{\sim}$.

The study of the separating and the completely separating collections has been quite useful to the kernel theory (see, e.g., [4])*. In fact, the separation condition in Theorem 2.7 simply states that $\mathcal{E}^i_{T^i_j} \equiv \{S \cap T^i_j | S \in \mathcal{E}^i\}$ is separating over the equivalence class T^i_j. A particular case of this observation is:

LEMMA 5.5: If Γ is a monotonic game with a non-negative characteristic function and if $x \in PX(\Gamma)$ then $\mathcal{D}(x)$ (see 4.16)) is a separating collection.

It will be convenient to associate with a collection $\mathcal{E} = \{S_1, S_2, \ldots, S_\alpha\}$ of subsets of N the characteristic vectors $\chi^{S_1}, \chi^{S_2}, \ldots, \chi^{S_\alpha}$, where

$$(5.2) \qquad \chi_i^{S_\nu} = \begin{cases} 1 & \text{if } i \in S_\nu \\ 0 & \text{if } i \notin S_\nu, \end{cases} \qquad \nu = 1,2,\ldots,\alpha.$$

Definition 5.6: A collection $\mathcal{E} = \{S_1, S_2, \ldots, S_\alpha\}$ of subsets of a set N is called balanced, if positive constants $c_1, c_2, \ldots, c_\alpha$ exist such that

$$(5.3) \qquad \sum_{\nu=1}^{\alpha} c_\nu \chi^{S_\nu} = \chi^N.$$

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*See [9] and [12] for additional properties of separating collections.

$\mathcal{B}$ is called minimal balanced if it is balanced and none of its proper sub-collections is balanced.

$\mathcal{B}$ is called weakly balanced if (5.3) is satisfied by non-negative constants $c_1,\ldots,c_\alpha$. The constants $c_1,\ldots,c_\alpha$ are called balancing coefficients.

Balanced and minimal balanced collections were introduced* and studied in [1] and [13]. They are useful to the study of various solution concepts such as the core (see [13],[10]), the bargaining set (see [3]) and, as we shall see in this paper, the kernel. See [8] for additional information concerning their structure.

LEMMA 5.7: A balanced collection is separating.

The proof is straightforward. The converse statement, however, is not true. Indeed, any set of six minimal winning coalitions in the 7-person projective game is completely separating and not even weakly balanced. It turns out, however, that imposing a near-ring requirement is a remedy:

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*O. N. Bondareva uses the term (q-θ) - covering [reduced (q-θ) - covering] to denote the pair consisting of weights and a collection of characteristic vectors of a balanced [minimal balanced] collection (see [1]). It was convenient in [13] to rule out the collection {N}. This exception is not needed here.

THEOREM 5.8: A separating near-ring collection $\mathcal{C}$ of subsets of a set $N = \{1,2,\ldots,n\}$ which contains at least one non-empty subset is weakly balanced.

PROOF: There is no loss of generality in assuming that $\mathcal{C}$ is completely separating. It is immediately varified that the theorem holds for $n = 1$. We shall therefore also assume that $n \geq 2$. Let $\mathcal{C}_i \equiv \{S:S\in \mathcal{C}, i\notin S\}$. Let $\mathcal{C}_i^{\sim}$ denote the set of elements of $\mathcal{C}_i$ which are maximal under inclusion; we shall show that $\mathcal{C}_i^{\sim}$ is a partition of $N - \{i\}$. Indeed, it follows from the complete separating property that each member of $N - \{i\}$ belongs to at least one element of $\mathcal{C}_i$. By the near-ring property, the elements of $\mathcal{C}_i^{\sim}$ are disjoint. The collection $\mathcal{C}^{\sim} \equiv \mathcal{C}_1^{\sim} \cup \mathcal{C}_2^{\sim} \cup \ldots \cup \mathcal{C}_n^{\sim}$ is balanced. In fact, if $c(S)$ is the number of elements i such that $S\in \mathcal{C}_i^{\sim}$, then $c(S)/(n-1)$ is a balancing coefficient for S in $\mathcal{C}^{\sim}$. this completes the proof.

6. The Kernel and the Pseudo Kernel in Convex Games.

Let $\Gamma = (N;v)$ be an n-person game whose characteristic function merely satisfies

$$(6.1) \qquad v(N) \geq v(\{1\}) + \ldots + v(\{n\}), \quad v(\emptyset) = 0.$$

An n-tuple $(x_1,x_2,\ldots,x_n)$ satisfying

$$(6.2) \qquad x_i \geq v(\{i\}),\ i = 1,2,\ldots,n, \qquad x(N) = v(N)$$

will be called an imputation.

An imputation x is said to belong to the kernel $K = K(\Gamma)$ of Γ (for the grand coalition), if (see(2.7))

$$(6.3) \qquad s_{k,\ell}(x) \leq s_{\ell,k}(x) \quad \text{or} \quad x_\ell = v(\{\ell\}) \quad \text{for all } k,\ell \in N,\ k \neq \ell.$$

Note that, unlike the pseudo-kernel, the kernel is a relative invariant under strategic equivalence. For this reason, the kernel is more interesting from the game theoretic point of view. Note, however, that if $v(\{i\}) = 0$, $i = 1,2,\ldots,n$, then the kernel and the pseudo kernel coincide. For this reason, it is often possible to use the structure of the pseudo-kernel in order to obtain properties of the kernel.*

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*The need to pass to the pseudo-kernel often stems from the fact that Theorem 3.6 and various variants of it are not true if PK is replaced by K.

LEMMA 6.1: If $\Gamma \equiv (N;v)$ is a game with a non-negative characteristic function then

(6.4) $\quad \rho\mathcal{K}(\Gamma) \cap \{x | x_i \geq v(\{i\}),\ i = 1,2,\ldots,n\} \subset \mathcal{K}(\Gamma).$

PROOF: Compare (6.3) with (2.7).

THEOREM 6.2: If $\Gamma \equiv (N;v)$ is a convex game with a non-negative characteristic function then

(6.5) $\quad \rho\mathcal{K}(\Gamma) = \mathcal{K}(\Gamma).$

PROOF: Let $x \in \rho\mathcal{K}(\Gamma)$. It has been proved in* [6] that x belongs to the core of the game. Thus, by (3.2) and (6.4), $x \in \mathcal{K}(\Gamma)$. Conversely, let $x \in \mathcal{K}(\Gamma)$ and let $\Gamma^* \equiv (N;v^*)$ be a game which is strategically equivalent to Γ, for which $v^*(\{i\}) = 0$, $i = 1,2,\ldots,n$. Let x^* be the corresponding payoff. Clearly, $x^* \in \mathcal{K}(\Gamma^*) = \rho\mathcal{K}(\Gamma^*)$, because the kernel is a relative invariant under strategic equivalence. Moreover, Γ^* is a convex game because convexity is invariant under strategic equivalence. In particular Γ^* is monotonic and therefore (see (2.8)a) $s^*_{k,\ell}(x^*) = s^*_{\ell,k}(x^*)$ for all k,ℓ, $k \neq \ell$, where stars refer to entities with respect to Γ^*. The latter equalities, however, imply $s_{k,\ell}(x) = s_{\ell,k}(x)$ for all k,ℓ, $k \neq \ell$. Consequently $x \in \rho\mathcal{K}(\Gamma)$.

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*Theorems 2.4 and 2.11 of [6] actually refer to the kernel, but ommitting the requirement $v(\{i\}) = 0$, $i = 1,2,\ldots,n$, which appears in the proofs make them valid proofs for the pseudo-kernel.

Theorem 6.2 enables us to use results concerning the pseudo-kernels of convex games with non-negative characteristic functions in order to deduce properties of the kernels of such games. In as much as such properties are invariant under strategic equivalence they will remain true also for convex games which clearly satisfy (6.1).

7. The Kernel of a Convex Game Consists of a Unique Point. Characterizing its Location.

As stated at the end of the previous section, the first statement in the title of this section will be proved if we show:

THEOREM 7.1: The pseudo-kernel (for the grand coalition) of a convex game with a non-negative characteristic function consists of a unique point.

PROOF: Let $x,y \in PK(\Gamma)$. Denote

$$(7.1) \qquad \mathcal{D}(x) = \{S:S \neq N.\emptyset,\ e(S,x) \geq e(P,x) \quad \text{whenever } P \neq \emptyset,N\}$$

$$(7.2) \qquad \mathcal{D}(y) = \{R:R \neq N.\emptyset,\ e(R,y) \geq e(P,y) \quad \text{whenever } P \neq \emptyset,N\}$$

$$(7.3) \qquad s(x) = e(S,x),\ S \in \mathcal{D}(x)$$

$$(7.4) \qquad s(y) = e(R,y),\ R \in \mathcal{D}(y)$$

Without loss of generality one may assume that

$$(7.5) \qquad s(x) \leq s(y).$$

The theorem certainly holds for 1-person and 2-person games. We shall proceed by induction, assuming that $n \geq 3$.

A. $\mathcal{D}(y)$ is a separating near-ring (Lemmas 5.2 and 5.5) which contains a non-empty subset of N; hence it contains a balanced collection $\mathcal{R} \equiv \{R_1, R_2, \ldots, R_\alpha\}$ (Theorem 5.8). If $R_j \notin \mathcal{D}(x)$ then $e(R_j,x) < s(x) \leq s(y) = e(R_j,y)$. Consequently $x(R_j) > y(R_j)$. If $R_j \in \mathcal{D}(x)$ then we can only conclude that $x(R_j) \geq y(R_j)$. Multiplying these inequalities by the balancing coefficients, we obtain $x(N) \geq y(N)$, with equality occuring only if $\mathcal{R} \subset \mathcal{D}(x)$ and $s(x) = s(y)$. But equality must occur because $x(N) = v(N) = y(N)$; hence we conclude that there exists a set R in $\mathcal{D}(x) \cap \mathcal{D}(y)$ and, moreover,

$$(7.6) \qquad x(R) = y(R). \qquad x(N-R) = y(N-R).$$

B. Let $m(x) + 1$ be the last stage of the profile of x and let $m(y) + 1$ be the last stage of the profile of y. Consider the stage games $(T^*_R; v^*_R)$ and $(T^*_{N-R}; v^*_{N-R})$ as defined in Corollary 4.7, with respect to x and for $i = m(x) + 1$. Consider also the analogous stage games $(T^{**}_R; v^{**}_R)$ and $(T^{**}_{N-R}; v^{**}_{N-R})$ with respect to y and for $i = m(y) + 1$. The players in these stage games are 1-element sets.

Since x and y belong to $PK(\Gamma)$, they a fortiori belong to the core of Γ (see the proof of Theorem 6.2). By Theorem 4.1, all these games are convex and have a non-negative characteristic function; consequently they are monotonic. Without loss of generality we may assume that $R = \{1,2,\ldots,r\}$. We can now use Theorem 3.6 to conclude that

$$(7.7) \qquad (x_1,\ldots,x_r) \in \rho K(T_R^*;\ v_R^*),$$
$$(x_{r+1},\ldots,x_n) \in \rho K\ (T_{N-R}^*;\ v_{N-R}^*),$$

$$(7.8) \qquad (y_1,\ldots,y_r) \in \rho K(T_R^{**};\ v_R^{**}),$$
$$(y_{r+1},\ldots,y_n) \in \rho K(T_{N-R}^{**};\ v_{N-R}^{**}).$$

However, because of (7.6) and Corollary 4.7, $(T_R^*;\ v_R^*)$ and $(T_R^{**};\ v_R^{**})$ are the same game since they have the same set of players and the same characteristic function. Similarly, $(T_{N-R}^*;\ v_{N-R}^*)$ and $(T_{N-R}^{**};\ v_{N-R}^{**})$ are the same game. The same games possess the same pseudo-kernel. Since all these games have fewer than n players, we can use the induction hypothesis to conclude that their pseudo-kernels consist of unique points. Consequently, by (7.7) and (7.8), $x = y$. This completes the proof in view of the fact that the pseudo-kernel is not empty (see [4]).

Corollary 7.2: The kernel (for the grand coalition) of a convex game consists of a unique point.

Corollary 7.2 brings to an end the main purpose of this study. We now know exactly the shape of the kernel of a convex game; namely, a unique point. There remains, however, the problem of locating this point - preferably in geometrical terms. Fortunately, general theorems are available in the literature which will enable us to complete our task: In [11], D. Schmeidler introduced the nucleolus of the game

and proved that it is a subset of the kernel consisting of a unique point. In view of this result and Corollary 7.2 we can now state:

Corollary 7.3: The kernel and the nucleolus of a convex game (for the grand coalition) coincide.

In [7], we present a characterization of the nucleolus for a general cooperative game. For games with a non-empty core it coincides with the lexicographic core defined as follows:

Let $\Gamma_0 \equiv \Gamma \equiv (N;v)$ be an n-person game, $n \geq 2$, whose core is not empty. We shall construct games $\Gamma_1 \equiv (N;v_1)$, $\Gamma_2 \equiv (N;v_2),\ldots,$ $\Gamma_m \equiv (N;v_m)$ over the same set of players N, whose characteristic functions are defined inductively by

$$(7.9)\qquad v_i(S) = \begin{cases} v_{i-1}(S) & \text{if } S \in \mathcal{A}(\Gamma_{i-1}) \\ v_{i-1}(S) + \delta_i & \text{if } S \notin \mathcal{A}(\Gamma_{i-1}). \end{cases}$$

Here,

$$(7.10)\qquad \mathcal{A}(\Gamma_{i-1}) \equiv \{S \mid e(S,x) = 0 \text{ whenever } x \text{ belongs to the core of } \Gamma_{i-1}\}$$

and δ_i is maximal under the requirement that Γ_i has a non-empty core. The last game Γ_m is characterized by the requirement that it is the first game in the sequence which is inessential.

Definition 7.4: The lexicographic core of Γ is the core of Γ_m.

Note that the lexicographic core is a point whose location can be described in geometric terms, regarding the core as a fundamental set. Its location can be obtained by "pushing inside" at equal ℓ_1-distances the appropriate hyperplanes which determine the core* - stopping the push of a hyperplane only when it causes the inside to become empty.

By Corollary 7.3, and since a convex game has a non-empty core, we can finally state:

THEOREM 7.5: The kernel of a convex game coincides with the lexicographic core.

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*Another way of putting it: "pushing inside" these hyperplanes in such a way that their intersections with the axes indexed by the members of the corresponding coalitions move equal distances.

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